An Aharonov-Bohm-like effect for simply connected regions arising due to boundary conditions

1989 J. Phys. A: Math. Gen. 22 L425

(http://iopscience.iop.org/0305-4470/22/10/002)

J. Phys. A: Math. Gen. 22 (1989) L425-L430. Printed in the UK

LETTER TO THE EDITOR

An Aharonov–Bohm-like effect for simply connected regions arising due to boundary conditions

S M Roy and Virendra Singh

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Received 24 October 1988

Abstract. We demonstrate the non-zero effect of the electromagnetic vector potential on charged particles confined to field-free simply connected regions in which no non-zero gauge-invariant loop integrals of the vector potential exist. This effect arises in the presence of certain boundary conditions. We propose an experimental test of this effect.

In classical electromagnetism a charged particle is influenced only by the electric and magnetic fields at the location of the particle. In quantum mechanics also the Heisenberg equations of motion depend locally on fields. Yet Aharonov and Bohm [1] claimed that for particles in multiply connected regions in quantum mechanics vector potentials or inaccessible fields have an increased significance (AB effect). Wu and Yang [2] therefore concluded that field strengths underdescribe electromagnetism and non-integrable phase factors $\exp[ie \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}/(\hbar c)]$ for all closed loops correctly describe electromagnetism.

We demonstrate here for a variety of commonly used boundary conditions that electromagnetic vector potentials affect charged particles confined to field-free regions even when the region is simply connected so that no non-zero gauge-invariant loop integrals of the vector potential exist. This means that for given potential independent boundary conditions, two potentials $A^{(1)}$ and $A^{(2)}$ which yield the same fields everywhere in accessible and inaccessible regions may lead to different effects quantum mechanically. These boundary conditions bring out a new facet of the idea of gauge invariance just as in the AB effect the single-valuedness boundary condition has been said to display increased significance of vector potentials or the effect of inaccessible fields [1, 2].

However, it was proposed by Bocchieri and Loinger [3] in the time-independent AB situation that potential dependent quasiperiodic boundary conditions may compensate for the effect of inaccessible fields.

Nevertheless when the inaccessible flux is changed from one constant value at $t < t_1$ to another for $t > t_2$, it has been shown [4] that the boundary conditions initially and finally must be the same, i.e. independent of the potential (or inaccessible flux). Hence a non-zero AB effect of changing the inaccessible flux results. It is similarly proved for simply connected situations here that in the time-dependent case the boundary conditions must be independent of the time-varying potential, and hence a non-zero effect is predicted. An experimental test of the effect and an interpretation in terms of accessible fields similar to that suggested for the AB effect [5] will be proposed.

0305-4470/89/100425+06$02.50

The first example we consider is that of a toroidally confined magnetic flux. This is because any finite straight solenoid produces potential which can be expressed in terms of fields outside the solenoid only, raising ambiguities in interpretation [6]. Indeed, beautiful experimental evidence concerning the AB effect in the toroidal situation has already been obtained [7]. We shall come back later to examples involving an infinite straight solenoid.

Consider a magnetic field configuration symmetric about the z axis directed everywhere in the azimuthal (ϕ) direction and non-zero only in the region between two coaxial cylinders of radii a_1 and a_2, each extending from $z=-b_1$ to $z=b_1$ (figure 1). In cylindrical coordinates, with $\hat{\phi}$ the unit vector in the ϕ direction,

$$\hat{B} = \hat{\phi}\theta(a_2-\rho)\theta(\rho-a_1)\theta(b_1-z)\theta(b_1+z)F/[2b_1(a_2-a_1)] \quad (1)$$

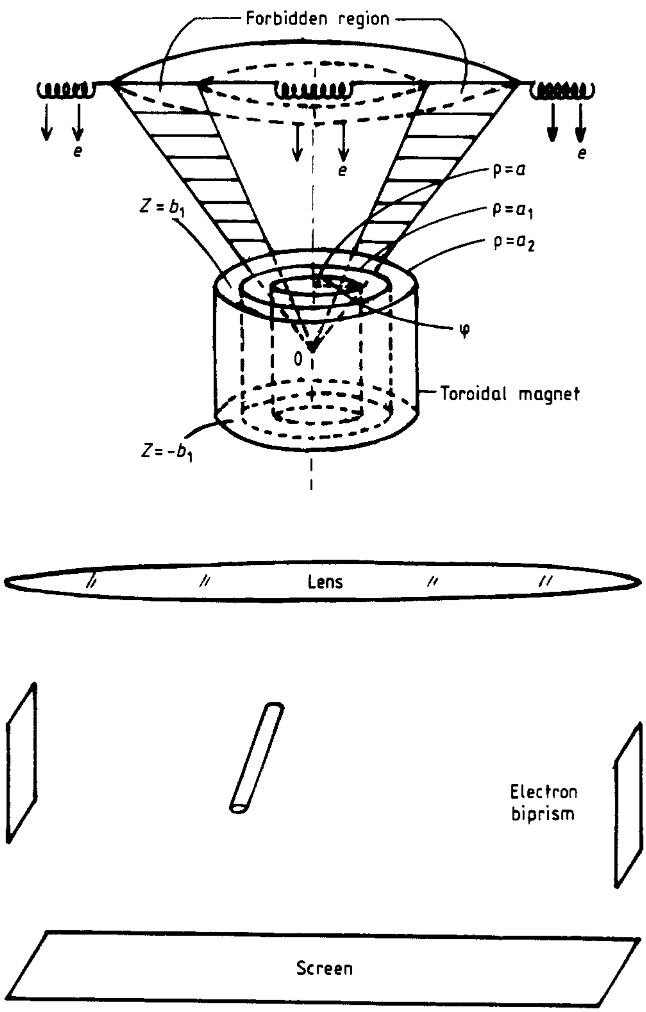

Figure 1. Electron interference experiment with extra forbidden region (shaded) to make electron paths lie in a simply connected region. The effect of changing flux in the toroidal magnet should be investigated experimentally for various angular sizes of the extra forbidden region.

where $\theta(x) \equiv 1$ for $x \geqslant 0$, $\theta(x) \equiv 0$ for $x < 0$, and F is the total magnetic flux linked to any semi-infinite ρ-z plane with the z axis as boundary. Examples of vector potentials which give this magnetic field are

$$A^{(1)} = \hat{z}[(a_2 - \rho)\theta(a_2 - \rho) - (a_1 - \rho)\theta(a_1 - \rho)]\theta(b_1 - z)\theta(b_1 + z)f \qquad (2)$$

$$A^{(2)} = \hat{z}[(\rho - a_2)\theta(\rho - a_2) - (\rho - a_1)\theta(\rho - a_1)]\theta(b_1 - z)\theta(b_1 + z)f \qquad (3)$$

where

$$f \equiv F/[2b_1(a_2 - a_1)]. \qquad (4)$$

We shall consider in succession two different physical systems interacting with vector potentials gauge equivalent to $A^{(1)}$ and $A^{(2)}$. We always choose $A_0 = 0$.

(i) *Non-relativistic particle of charge e confined to the cylindrical region* $|z| \leqslant b < b_1$, $\rho \leqslant a < a_1$. In this field-free region consider gauge choices $A_\rho = A_\phi = 0$, A_z independent of ρ, ϕ and obeying the periodicity condition $A_z(z = b) = A_z(z = -b)$ (e.g. $A^{(1)} = \hat{z}(a_2 - a_1)f$, $A^{(2)} = 0$). The time-independent Schrödinger equation has the form ($\hbar = c = 1$, $M =$ particle mass),

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \left(\frac{\partial}{\partial z} - \mathrm{i}eA_z\right)^2\psi + k^2\psi = 0 \qquad (5)$$

where $k^2 = 2ME$. We first assume periodic boundary conditions in ϕ and z (i.e. $\psi(z = b) = \psi(z = -b)$, $(\partial_z\psi)(z = b) = (\partial_z\psi)(z = -b)$ and similarly for ϕ) and Dirichlet boundary conditions $\psi = 0$ for $\rho = a$. Setting

$$\psi = \exp\left[\mathrm{i}m\phi + \mathrm{i}\pi nz/b + \mathrm{i}e\int_{-b}^{z} \mathrm{d}z'\, A_z - \mathrm{i}ez\int_{-b}^{b} \mathrm{d}z'\, A_z/(2b)\right]\psi_{mn}(\rho) \qquad (6)$$

where m and n are integers, we find that ψ_{mn} obeys Bessel's equation whose solution regular at the origin $\rho = 0$ is

$$\psi_{mn} = \text{constant} \times J_m(k'\rho) \qquad (7)$$

where

$$k'^2 = k^2 - \left(\frac{n\pi}{b} - \frac{e}{2b}\int_{-b}^{b} \mathrm{d}z'\, A_z(z')\right)^2. \qquad (8)$$

The condition $\psi_{mn} = 0$ for $\rho = a$ yields

$$k' = x_{rm}/a \qquad J_m(x_{rm}) = 0$$

where x_{rm} are the zeros of $J_m(x)$. The energy eigenvalues are

$$E = \frac{1}{2M}\left[\left(\frac{x_{rm}}{a}\right)^2 + \left(\frac{n\pi}{b} - \frac{e}{2b}\int_{-b}^{b} \mathrm{d}z'\, A_z(z')\right)^2\right] \qquad (9)$$

which are clearly gauge dependent. For example, the two potentials $A^{(1)}$ and $A^{(2)}$ give the same field everywhere (inside and outside the cylinder where the particle is confined) but different energy eigenvalues.

It may be tempting to ascribe this Aharonov-Bohm-like effect in a simply connected region to a mathematical multiple connectivity introduced by the periodic boundary condition. We shall, however, show that a similar effect exists also for the Neumann boundary conditions $\partial\psi/\partial z = 0$ at $z = \pm b$, which moreover may be easier to achieve experimentally than the periodic. Keeping boundary conditions in ρ and ϕ the same as before we obtain the (non-periodic) energy eigenfunctions

$$\psi = \text{constant} \times \exp\left[im\phi + ie \int_{-b}^{z} dz' A_z \right] J_m(x_{rm}\rho/a)$$
$$\times \left[\left\{ \left(eA_z(b) - \frac{n\pi}{2b} \right) \exp(in\pi(z+b)/(2b)) \right\} - \{n \to -n\} \right]$$

where m, n are integers and $J_m(x_{rm}) = 0$. The energy eigenvalues $E = [(x_{rm}/a)^2 + (n\pi/(2b))^2]/(2M)$ are independent of the potential. There *is*, however, an effect of the potential since $|\psi|$ depends on $A_z(b)$ and is not proportional to the eigenfunction for zero vector potential.

Now, in the spirit of Bocchieri and Loinger [3], one may attempt to compensate for this effect by replacing the boundary conditions we used by potential dependent boundary conditions at $z = \pm b$. For example, replacing periodic boundary conditions by

$$\{\psi, (\partial_z \psi)\}_{z=b} = \{\psi, \partial_z \psi\}_{z=-b} \exp\left(ie \int_{-b}^{b} dz' A_z(z') \right) \tag{10}$$

and Neumann boundary condition by $(\partial/\partial z - ieA_z)\psi = 0$ at $z = \pm b$, we would obtain $|\psi|$ and energy eigenvalues independent of the potential.

The analogous attempt [3] to remove the AB effect has already received the following objection [4] which applies equally well in the present case. In a time-dependent situation in which the potential is changed smoothly from zero to some final non-zero value staying periodic (single valued) at each time and corresponding finally to zero fields in the allowed region, the wavefunction remains periodic (single valued) if it is periodic at $t = 0$ because the Hamiltonian does not destroy periodicity [4]. Adapted to our case, this means that if the boundary condition (10) holds initially, it cannot hold finally. For example, if A_z is independent of z and a function of time, zero at $t = 0$, and ψ is periodic in z at $t = 0$, then an exact solution of the time-dependent Schrödinger equation is given by

$$\psi = \text{constant} \times \exp\left(im\phi + i\pi n \frac{z}{b} - i \int_0^t dt'\, E(t') \right) J_m(x_{rm}\rho/a)$$

which is periodic in z at all times. Here $E(t')$ is calculated from the instantaneous value of A_z using (9). A similar explicit calculation for general boundary conditions is difficult for arbitrary time dependence of $A_z(t)$; but if $A_z(t) = \Sigma_i c_i \theta(t_i)$, we see by integrating the time-dependent Schrödinger equation in time that ψ is continuous at $t = t_i$. Hence ψ obeys the same spatial boundary conditions for $t > t_i$ as for $t < t_i$, and consequently also for all time. Hence the effect of potentials found here cannot be removed by using potential dependent boundary conditions in the time-dependent case.

(ii) *Relativistic Dirac particle of charge e confined to the region* $|z| \leq b < b_i$ *of the z axis* ($\rho = 0$). The problem of a Dirac fermion in a one-dimensional box interacting with a scalar solitonic potential was considered earlier with periodic [8] as well as with more general boundary conditions [9] to elucidate the phenomenon of fractional fermion number [10]. Here we set the scalar potential to zero and include the vector potential $A^{(1)}$ or $A^{(2)}$ which give zero field on the z axis:

$$H = -i\sigma_2 \left(\frac{\partial}{\partial z} - ieA_z \right) \tag{11}$$

where σ_2 is the usual Pauli matrix. The 2×2 wavefunction ψ is conveniently expressed as

$$\psi=\psi_+(z)e_+ + \psi_-(z)e_- \tag{12}$$

where $e_\pm$ are eigenfunctions of σ_2, with eigenvalues ±1.

The Hamiltonian is self-adjoint for a four-parameter family of boundary conditions

$$\begin{pmatrix}\psi_+(b)\\ \psi_-(-b)\end{pmatrix}=U\begin{pmatrix}\psi_-(b)\\ \psi_+(-b)\end{pmatrix}$$

where U is a 2×2 unitary matrix characterised by four real parameters α, β, γ, λ:

$$U=\mathrm{e}^{\mathrm{i}\alpha}\begin{pmatrix}\cos\lambda\ \mathrm{e}^{\mathrm{i}\beta} & \sin\lambda\ \mathrm{e}^{\mathrm{i}\gamma}\\ \sin\lambda\ \mathrm{e}^{-\mathrm{i}\gamma} & -\cos\lambda\ \mathrm{e}^{-\mathrm{i}\beta}\end{pmatrix}.$$

The energy eigenvalues are easily seen to be

$$E=(2n\pi+\alpha\pm\zeta)/(2b)$$

where n is any integer and where

$$\cos\zeta\equiv\sin\lambda\ \cos(\gamma-eA_z2b).$$

Evidently the energy eigenvalues depend on A_z whenever $\sin\lambda\neq0$. Attempts to remove the effect face similar problems to those in case (i).

(iii) *Particle confined to the region* $0\leqslant\phi\leqslant\phi_0<2\pi$ *of a ring* $\rho=a$, $z=0$ *outside an infinite straight solenoid.* The Schrödinger equation with constant potentials A_ϕ and periodic boundary conditions $\psi(\phi=0)=\psi(\phi=\phi_0)$, $(\partial_\phi\psi)(0)=(\partial_\phi\psi)(\phi_0)$ gives energy eigenfunctions with gauge-dependent eigenvalues

$$\psi_m=\text{constant}\times\exp(\mathrm{i}m\phi2\pi/\phi_0)\qquad E_m=\frac{(2\pi)^2}{2Ma^2\phi_0^2}\left(m-\frac{ea\phi_0}{2\pi}A_\phi\right)^2 \tag{13}$$

where m is an integer. The Neumann boundary conditions reveal dependence of $|\psi_m|$ on the potential.

With suitable changes in [1, 11] one could also compute scattering by an infinitely long solenoid when charged particles are confined to the region $0\leqslant\phi\leqslant\phi_0<2\pi$ outside the solenoid.

(iv) *Gauge-invariant interpretation of the effect and a proposed experimental test.* In the time-independent case both the AB effect and the present effect can be compensated for by using potential dependent boundary conditions. Nevertheless when inaccessible fluxes are changed, in the class of gauges with zero scalar potential, the change in the vector potential

$$\Delta\boldsymbol{A}=\int_{t_1}^{t_2}\mathrm{d}t\,\dot{\boldsymbol{A}}=-c\int_{t_1}^{t_2}\mathrm{d}t\,\boldsymbol{E}$$

is a gauge-invariant quantity and can cause physical effects because the time-dependent Schrödinger equation does not allow the boundary condition to change with the potential. Such an interpretation has already been suggested [5] for the AB effect and applies equally well to the present effect.

We thus see from the present examples, that (a) electromagnetic potentials have a significance in quantum mechanics even in systems living in simply connected regions for fixed boundary conditions. This does not violate gauge invariance because (b) consideration of time evolution allows us to interpret them in terms of the changing electric fields in the accessible region during the period of establishing these vector potentials.

We propose now an experimental test. In figure 1 we show schematically an electron interference experiment with an extra forbidden region (shaded) added such that the electron paths lie in a simply connected region. The obstacles introduced may be highly reflecting to achieve Neumann boundary conditions. The forbidden region lies between two cones with apex at the centre of a toroidal magnet and is outside the torus. Apart from this crucial modification, the experimental set-up is that used by Tonomura *et al* [7] for their test of the AB effect. With the electron beam switched on, the experiment should aim to detect a change in the interference pattern when the magnetic flux through a toroidal magnet is changed, and to detect the dependence of the effect on the angular size of the forbidden region. A similar modification of the Möllenstedt and Bayh experiment [12] in which a part of the azimuthal region outside a straight solenoid is blocked in order to make the accessible region simply connected should also be tried.

One of us (SMR) thanks Sudhendu Rai Choudhury and Ashoke Sen for very fruitful conversations. We also wish to thank the referee for informing us of a recently discovered geometrical quantum phase effect due to localised changes in spatial boundary conditions which can be tested by a neutron interference experiment [13].

References

[1] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[2] Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3843, 3845
Yang C N 1984 *Proc. Int. Symp. on Foundations of quantum mechanics, Tokyo, 1983* (*J. Phys. Soc. Japan*)
[3] Bocchieri P and Loinger A 1978 *Nuovo Cimento* A **47** 475; 1980 *Nuovo Cimento* **59** 121; 1979 *Lett. Nuovo Cimento* **25** 476
Boccheiri P, Loinger A and Siragusa G 1979 *Nuovo Cimento* A **51** 1; 1980 *Nuovo Cimento* **56** 55
Miyazawa H 1984 *Proc. Int. Symp. on Foundations of quantum mechanics, Tokyo, 1983* (*J. Phys. Soc. Japan*)
Yang C N 1984 *Proc. Int. Symp. on Foundations of quantum mechanics, Tokyo, 1983* (*J. Phys. Soc. Japan*)
[4] Peshkin M 1981 *Phys. Rev.* A **23** 360; *Phys. Rep.* **80** 375
Wilczek F 1982 *Phys. Rev. Lett.* **48** 1144
Aharonov Y 1984 *Proc. Int. Symp. on Foundations of quantum mechanics, Tokyo, 1983*, (*J. Phys. Soc. Japan*)
[5] Roy S M and Singh V 1984 *Nuovo Cimento* **79A** 391
Goldin G A and Simon B *The Aharonov–Bohm effect and Faraday's law of induction* (unpublished)
Yamada M 1987 *Nuovo Cimento* **98A** 205
[6] Roy S M 1980 *Phys. Rev. Lett.* **44** 111
Klein U 1981 *Phys. Rev.* D **23** 1463
Lipkin H J 1981 *Phys. Rev.* D **23** 1466
[7] Tonomura A *et al* 1982 *Phys. Rev. Lett.* **48** 1443; 1986 *Phys. Rev. Lett.* **56** 792; 1986 *Phys. Rev.* A **34** 815
Kupor C G 1980 *Phys. Lett.* **79A** 413
[8] Kivelson S and Schrieffer J R 1982 *Phys. Rev.* B **25** 6447
Rajaraman R and Bell J S 1982 *Phys. Lett.* **116B** 151
[9] Roy S M and Singh V 1984 *Phys. Lett.* **143B** 179
[10] Jackiw R and Rebbi C 1976 *Phys. Rev.* D **13** 3398
[11] Ruijsenaars S N M 1983 *Ann. Phys., NY* **146** 1
[12] Möllenstedt G and Bayh W 1961 *Naturwiss* **48** 400; 1962 *Naturwiss* **49** 81
[13] Lévy-Leblond J-M 1987 *Phys. Lett.* **25A** 441
Greenberger D M 1988 *Physica* B **151** 374